Journal of Engineering Mathematics **49:** 305–319, 2004.
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Green's functions and three-dimensional steady-state heat-conduction problems in a two-layered composite

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Received 5 December 2003; accepted in revised form 29 March 2004

Abstract. A boundary-value problem for steady-state heat conduction in a three-dimensional, two-layered composite is studied. The method of Green's function is used in the study. Green's functions are constructed as double sums in terms of eigenfunctions in two of the three directions. The eigenfunctions in the direction orthogonal to the layers are unconventional and must be defined appropriately. The use of different forms of the Green's functions leads to different representations of the solutions as double sums with different convergence characteristics and it is shown that the method of Green's functions is superior to the classical method of separation of variables.

Key words: analytical solutions, Green's functions, heat conduction, two-layered composite

1. Introduction

Computer codes are often developed using the finite-element, finite-difference or boundary-element methods for finding numerical solutions of engineering problems. In the verification of such numerical codes, methods that are capable of generating numerical solutions of high accuracy, say with at least ten place accuracy, are needed. The present paper aims at providing such a method for steady-heat-conduction problems. More specifically, let us consider a two-layered composite occupying the regions $0 < x < a, 0 < z < d$, and $-b < y < 0$ and $0 < y < c$ respectively. Each of the layers is assumed isotropic. The faces $x = 0, x = a, z = 0$, and $z = d$ are subject to boundary conditions of the first or the second kind, while the faces $y = -b$ and $y = c$ are subject to boundary conditions of the first, the second, or the third kind. By superposition it suffices to consider the case where only one of the six faces is subject to a non-homogeneous boundary condition. Both perfect and imperfect thermal interf ace conditions at $y = 0$ are considered.

Our main goal in this paper is to present the method of Green's functions for layered composites as a method of high precision. We shall illustrate through the example how solutions of boundary-value problems in steady-state heat conduction can be constructed using the method of Green's functions. Background materials for the method of Green's functions can be found in [1], which deals with homogenous bodies. In this paper we shall define Green's functions for layered composite materials and construct them as double series using the one-dimensional eigenfunctions. We note that the eigenfunctions in the direction orthogonal to the layers is unconventional and must be defined and treated carefully. Each Green's function may be constructed as double series in three ways, by using eigenfunctions in two of the three directions. This leads to three different representation of the solution with its own distinct convergence characteristics and complementary properties. Rapid convergence is expected in

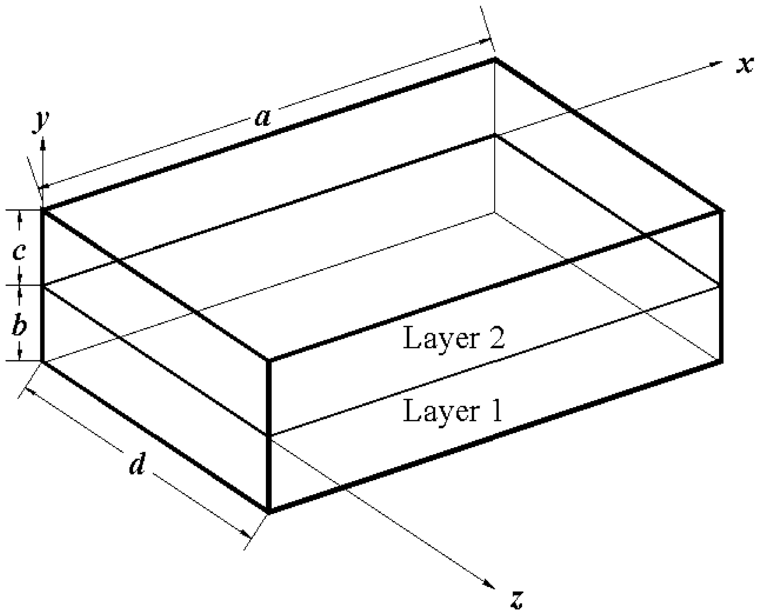

Figure 1. A two-layered composite.

general, except possibly at or near certain points on the boundary or the interface. It is seen that, just as in the case of homogeneous bodies, the method of Green's functions offers a useful and practical alternative to the classical method of separation of variables for problems with composite materials as well.

Solutions of heat conduction problems in layered composites are important in the design of modern engineering devices and there have been several studies devoted to such topics. Kennedy [2] presented analytical solutions for the axisymmetric temperature distribution for a cylinder with a small circular surface area heated on one end. The recent work by Haji-Sheikh, Beck and Agonafer [3] deals with multi-dimensional multi-layer bodies that complements the work on transient heat-conduction problems in multi-dimensional layered materials by Haji-Sheikh and Beck [4]. The work in [3] is based on the classical method of separation of variables and presents highly accurate numerical results.

The paper is organized as follows. In Section 2 we introduce the mathematical problem A and the associated Green's functions. We mention that Problem A here merely serves as an example problem and the method introduced here is capable of treating more general boundary value problems as we shall point out in the paper. In Section 3 we derive the representation of solutions using the Green's functions. The Green's functions are constructed in different forms using different choices of the spatial eigenfunctions. The eigenfunctions are studied in Section 4. In particular, the eigenfunctions in the direction orthogonal to the layers, *i.e.*, the y-direction, is unconventional and is treated in Section 4. In Section 5, we construct the first form of the Green's function using the x- and the z-eigenfunctions. In Section 6 we present the solution to Problem A using the first form of the Green's function. In Section 7 we present further solutions to Problem A using alternative forms of the Green's functions. Section 8 contains the discussion.

2. Mathematical problem and Green's functions

We shall consider the following mathematical problem in a two-layered composite in the regions below:

Layer 1: $0 < x < a,\ -b < y < 0,\ 0 < z < d,$ with thermal conductivity k_1 and temperature $T_1(x, y, z)$,

Layer 2: : $0 < x < a,\ 0 < y < c,\ 0 < z < d,$ with thermal conductivity k_2 and temperature $T_2(x, y, z)$.

The temperatures $T_i(x, y, z)$, $i = 1, 2$, satisfies the equation

$$L_i T_i(x, y, z) \equiv k_i \left(\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} + \frac{\partial^2 T_i}{\partial z^2} \right) = 0, \quad i = 1, 2. \tag{1}$$

where $L_i = k_i \nabla^2$. We have for Problem A the boundary conditions:

$$T_i = 0 \text{ at } x = 0,\ \frac{\partial T_i}{\partial x} = 0 \text{ at } x = a,\ \frac{\partial T_1}{\partial y} = 0 \text{ at } y = -b,$$
$$\frac{\partial T_2}{\partial y} = 0 \text{ at } y = c,\ T_i = 0 \text{ at } z = 0,\ k_i \frac{\partial T_i}{\partial z} = q_0 \text{ at } z = d, \tag{2}$$

and the interface conditions:

$$-k_1 \frac{\partial T_1}{\partial y}|_{y=0} = h(T_1 - T_2)|_{y=0},\ k_1 \frac{\partial T_1}{\partial y}|_{y=0} = k_2 \frac{\partial T_2}{\partial y}|_{y=0}, \tag{3}$$

where h is a contact conductance. Perfect contact results as $h \to \infty$.

According to [1] each specific Green's function and a specific geometry is identified with a number of the form XIJYKL in which X and Y represent the coordinate axes, and the letters following each axis name take on values 1, 2 or 3 to represent the type of boundary conditions present at the body faces normal that axis. For example, number X12 represents boundary conditions of type 1 at $x = 0$ and type 2 at $x = a$. We note that according to the numbering system in [1] Problem A has the designation X12B00Y2(C3)2B00Z12B01. For details on the numbering system see [1, Chapter 2]

The Green's function for Problem A above is a 2×2 matrix

$$g(x, y, z, x', y', z') = (g_{ij}), \quad i = 1, 2, \tag{4}$$

where the indices i and j refer, respectively, to where the observation point is and where the source point is. The Green's function satisfies the equation

$$\begin{pmatrix} k_1 \nabla^2 & 0 \\ 0 & k_2 \nabla^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = -\delta(x - x')\delta(y - y')\delta(z - z') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

along with the boundary conditions:

$$g_{ij}(\vec{x}, \vec{x}')|_{x=0} = 0,\ \frac{\partial g_{ij}}{\partial x}(\vec{x}, \vec{x}')|_{x=a} = 0,\ \frac{\partial g_{1j}}{\partial y}(\vec{x}, \vec{x}')|_{y=-b} = 0,$$
$$\frac{\partial g_{2j}}{\partial y}(\vec{x}, \vec{x}')|_{y=c} = 0,\ g_{ij}(\vec{x}, \vec{x}')|_{z=0} = 0,\ k_i \frac{\partial g_{ij}}{\partial z}(\vec{x}, \vec{x}'|_{z=d} = 0, \tag{6}$$

and the interface conditions

$$-k_1 \frac{\partial g_{1j}}{\partial y}\Big|_{y=0} = h(g_{1j} - g_{2j})\Big|_{y=0}, k_1 \frac{\partial g_{1j}}{\partial y}\Big|_{y=0} = k_2 \frac{\partial g_{2j}}{\partial y}\Big|_{y=0}. \tag{7}$$

We shall now show how the solutions of the boundary-value problems can be represented in terms of the Green's function and boundary data.

3. Representation of solutions using Green's functions

We shall be concerned in this section with the representation of solutions of boundary-value problems in terms of the Green's functions and boundary data. Let the source be in layer 1. Replacing $\vec{x} = (x, y, z)$ by $\vec{x}' = (x', y', z')$ in Equation (1) leads to

$$L_i' T_i(\vec{x}') \equiv k_i\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) T_i(\vec{x}') = 0, i = 1, 2. \tag{8}$$

Interchanging $\vec{x}$ and $\vec{x}'$ in Equation (5) yields

$$L_1' g_{11}(\vec{x}', \vec{x}) = -\delta(\vec{x} - \vec{x}'), \tag{9}$$
$$L_2' g_{21}(\vec{x}', \vec{x}) = 0, \tag{10}$$
$$L_1' g_{12}(\vec{x}', \vec{x}) = 0, \tag{11}$$
$$L_2' g_{22}(\vec{x}', \vec{x}) = -\delta(\vec{x} - \vec{x}') \tag{12}$$

Multiply Equation (8) for $i = 1$ by g_{11}, Equation (9) by $T_1(\vec{x}')$, subtract, and integrate over layer 1. Also, multiply Equation (8) for $i = 2$ by g_{21}, Equation (10) by $T_2(\vec{x}')$, subtract, and integrate over layer 2. Adding these two integrals yields

$$\begin{aligned} &\int\int\int_{\text{layer1}} [g_{11} L_1' T_1(\vec{x}') - T_1 L_1' g_{11}(\vec{x}', \vec{x})] \mathrm{d}\vec{x}' \\ &+ \int\int\int_{\text{layer2}} [g_{21} L_2' T_2(\vec{x}') - T_2 L_2' g_{21}(\vec{x}')] \mathrm{d}\vec{x}' = T_1(x, y, z). \end{aligned} \tag{13}$$

Integrating the left-hand side above by parts, applying the interface and boundary conditions, and after much simplification, we obtain

$$\begin{aligned} T_1(x, y, z) = &\int_{-b}^{0}\int_{0}^{a} g_{11}(x', y', z', x, y, z) k_1 \frac{\partial T_1}{\partial z'}\Big|_{z'=d} \mathrm{d}x' \mathrm{d}y' \\ &+ \int_{0}^{c}\int_{0}^{a} g_{21}(x', y', z', x, y, z) k_2 \frac{\partial T_2}{\partial z'}\Big|_{z'=d} \mathrm{d}x' \mathrm{d}y'. \end{aligned} \tag{14}$$

Similarly, by considering the case where the source is in layer 2, we have

$$\begin{aligned} T_2(x, y, z) = &\int_{-b}^{0}\int_{0}^{a} g_{12}(x', y', z', x, y, z) k_1 \frac{\partial T_1}{\partial z'}\Big|_{z'=d} \mathrm{d}x' \mathrm{d}y' \\ &+ \int_{0}^{c}\int_{0}^{a} g_{22}(x', y', z', x, y, z) k_2 \frac{\partial T_2}{\partial z'}\Big|_{z'=d} \mathrm{d}x' \mathrm{d}y'. \end{aligned} \tag{15}$$

Table 1. x- and z-eigenfunctions, eigenvalues and normalization constants.

Case	$u_m(x)$	β_m	N_{xm}	Range of m
X11	$\sin(\beta_m x)$	$m\pi/a$	$a/2$	1,2,...
X12	$\sin(\beta_m x)$	$(2m-1)\pi/2a$	$a/2$	1,2,...
X21	$\cos(\beta_m x)$	$(2m-1)\pi/2a$	$a/2$	1,2,...
X22	$\cos(\beta_m x)$	$m\pi/a$	$a(m=0), a/2(m>0)$	0,1,...
Case	$w_n(z)$	ν_n	N_{zn}	**Range of n**
Z11	$\sin(\nu_n z)$	$n\pi/d$	$d/2$	1,2,...
Z12	$\sin(\nu_n z)$	$(2n-1)\pi/2d$	$d/2$	1,2,...
Z21	$\cos(\nu_n z)$	$(2n-1)\pi/2d$	$d/2$	1,2,...
Z22	$\cos(\nu_n z)$	$n\pi/d$	$d(n=0), d/2(n>0)$	0,1,...

4. The *x*-, *z*- and *y*-eigenfunctions

As we mentioned above for Problem A there are three forms of Green's functions, constructed using two of the three sets of spatial eigenfunctions. More specifically, let $u_m(x)$ and $w_n(z)$ be the x- and z-eigenfunctions satisfying

$$\frac{d^2}{dx^2}u_m(x) + \beta_m^2 u_m(x) = 0, \tag{16}$$

$$\frac{d^2}{dz^2}w_n(z) + \nu_n^2 w_n(z) = 0 \tag{17}$$

with, respectively, the eigenvalues β_m^2 and ν_n^2 and the normalization constants N_{xm} and N_{zn}. The eigenfunctions, eigenvalues and the normalization constants depend on the x- and the z-boundary conditions and are given in Table 1. Table 1, in fact, contains four combinations of boundary conditions of eigenfunctions in the x- and z-directions. We consider now the y-eigenfunctions.

The eigenfunctions in the y-direction V_n have components v_{n1} and v_{n2} in layers 1 and 2, respectively. We write

$$V_n(y) = \begin{pmatrix} v_{n1}(y) \\ v_{n2}(y) \end{pmatrix} \tag{18}$$

and consider the eigenvalue problem for $V_n(y)$:

$$\frac{d^2}{dy^2}v_{n1}(y) + \mu_n^2 v_{n1}(y) = 0, \quad -b < y < 0, \tag{19}$$

$$\frac{d^2}{dy^2}v_{n2}(y) + \mu_n^2 v_{n2}(y) = 0, \quad 0 < y < c. \tag{20}$$

Here μ_n^2 (or μ_n)is the y-eigenvalue. We look for nontrivial solutions of Equations (19) and (20) above that satisfy the nine different combinations of boundary conditions at $y = -b$,

$y = c$ and the interface conditions at $y = 0$. The general solution of Equations (19) and (20) are

$$v_{n1}(y) = \overline{A}\sin(\mu_n y) + \overline{B}\cos(\mu_n y), \quad -b < y < 0, \tag{21}$$
$$v_{n2}(y) = \overline{C}\sin(\mu_n y) + \overline{D}\cos(\mu_n y), \quad 0 < y < c, \tag{22}$$

Applying the boundary conditions at $y = -b$ and $y = c$ leads to

$$\overline{A} = \zeta_1\overline{G},\ \overline{B} = \zeta_2\overline{G},\ \overline{C} = \omega_1\overline{H},\ \overline{D} = \omega_2\overline{H} \tag{23}$$

for some $\overline{G}$ and $\overline{H}$. The coefficients ζ_1, ζ_2, ω_1 and ω_2 are determined by the y-boundary conditions and are given in Table 2, where the dependence on n is suppressed. We have

$$\zeta_1 = -\sin(\mu_n b), \zeta_2 = \cos(\mu_n b), \omega_1 = \sin(\mu_n c), \omega_2 = \cos(\mu_n c). \tag{24}$$

Based on the interface conditions to be satisfied by T_1 and T_2 at $y = 0$ in Equation (7) and hence by $v_{n1}(y)$ and $v_{n2}(y)$ in Equations (21) and (22) we find that $\overline{A}$ and $\overline{B}$ are related to $\overline{C}$ and $\overline{D}$ by

$$M_3\begin{pmatrix}\overline{A}\\ \overline{B}\end{pmatrix} = M_4\begin{pmatrix}\overline{C}\\ \overline{D}\end{pmatrix}, \tag{25}$$

where

$$M_3 = \begin{pmatrix} k_1\mu_n/h & 1\\ 1 & 0\end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 1\\ \delta & 0\end{pmatrix}, \tag{26}$$

where $\delta = k_2/k_1$. Equation (25) above states that

$$M_3\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\overline{G} = M_4\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\overline{H}. \tag{27}$$

Using Equation (26) in Equation (27) we have

$$\begin{pmatrix}(k_1\mu_n\zeta_1)/h + \zeta_2\\ \zeta_1\end{pmatrix}\overline{G} = \begin{pmatrix}\omega_2\\ \delta\omega_1\end{pmatrix}\overline{H}. \tag{28}$$

For nontrivial solutions we must have

$$\delta k_1\zeta_1\omega_1\mu_n/h - \zeta_1\omega_2 + \zeta_2\delta\omega_1 = 0, \tag{29}$$

which is an equation for the eigenvalues μ_n. Equation (28) then determines nontrivial solutions for $\overline{G}$ and $\overline{H}$. Notice that as only the ratios $\overline{G}/\overline{H}$ are determined we may without loss of generality take $\overline{G}$ to be unity. The eigenfunctions are determined once $\overline{G}$ and $\overline{H}$ are determined.

In the special case when $h \to \infty$, Equation (29) reduces to

$$\zeta_1\omega_2 - \delta\zeta_2\omega_1 = 0. \tag{30}$$

For Problem A we have the quantities $\zeta_1, \zeta_2, \omega_1, and\ \omega_2$ given in Equation (24). Thus (30) now becomes

$$\sin(\mu_n b)\cos(\mu_n c) + \delta\cos(\mu_n b)\sin(\mu_n c) = 0. \tag{31}$$

Table 2. Coefficients $\zeta_1,\ \zeta_2,\ \omega_1,\ \omega_2$ for various y-boundary conditions; dependence of μ on n is suppressed.

Case	ζ_1	ζ_2	ω_1	ω_2
Y11	$\cos(\mu b)$	$\sin(\mu b)$	$-\cos(\mu c)$	$\sin(\mu c)$
Y12	$\cos(\mu b)$	$\sin(\mu b)$	$\sin(\mu c)$	$\cos(\mu c)$
Y13	$\cos(\mu b)$	$\sin(\mu b)$	$k_2\mu\sin(\mu c)$ $-h\cos(\mu c)$	$k_2\mu\cos(\mu c)$ $+h\sin(\mu_c)$
Y21	$-\sin(\mu b)$	$\cos(\mu b)$	$-\cos(\mu c))$	$\sin(\mu c)$
Y22	$-\sin(\mu b)$	$\cos(\mu b)$	$\sin(\mu c)$	$\cos(\mu c)$
Y23	$-\sin(\mu b)$	$\cos(\mu b)$	$k_2\mu\sin(\mu c)$ $-h\cos(\mu c)$	$k_2\mu\cos(\mu c)$ $+h\sin(\mu c)$
Y31	$-k_1\mu\sin(\mu b)$ $-h\cos(\mu b)$	$k_1\mu\cos(\mu b)$ $-h\sin(\mu b)$	$-\cos(\mu c)$	$\sin(\mu c)$
Y32	$-k_1\mu\sin(\mu b)$ $-h\cos(\mu b)$	$k_1\mu\cos(\mu b)$ $-h\sin(\mu b)$	$-\sin(\mu c)$	$\cos(\mu c)$
Y33	$-k_1\mu\sin(\mu b)$ $-h\sin(\mu_1 b)$	$k_1\mu\cos(\mu b)$ $-h\sin(\mu_1 b)$	$k_2\mu\sin(\mu c)$ $-h\cos(\mu c)$	$k_2\cos(\mu c)$ $+h\sin(\mu c)$

Finally, when $b = c$, Equation (31) above becomes

$$(1+\delta)\sin(\mu_n b)\cos(\mu_n b) = 0, \tag{32}$$

which yields $\mu_n = \frac{n\pi}{2b}$.

It follows that the eigenfunctions $V_n(y)$ are given by

$$\text{For n even: } v_{n1}(y) = \cos(\mu_n(y+b)),\ v_{n2}(y) = \cos(\mu_n(y-c)), \tag{33}$$

$$\text{For n odd: } v_{n1}(y) = \cos(\mu_n(y+b)),\ v_{n2}(y) = -\frac{1}{\delta}\cos(\mu_n(y-c)). \tag{34}$$

For $b \neq c$ or h being finite, closed-form expressions for the eigenvalues are not possible in general and one has to resort to numerical methods to obtain the eigenvalues and eigenfunctions in y. Efficient algorithms for handling numerical computations of eigenvalue problems such as the ones encountered here can be found in [5].

Let μ_n^2 and $V_n(y)$ form an eigenpair. Let μ_k^2 and $V_k(y)$ be another eigenpair. It can be shown that $V_n(y)$ and $V_k(y)$ are orthogonal for $\mu_n^2 \neq \mu_k^2$ in the inner product

$$\langle V_n(y), V_k(y)\rangle \equiv k_1\int_{-b}^{0} v_{n1}(y)v_{k1}(y)\mathrm{d}y + k_2\int_0^c v_{n2}(y)v_{k2}(y)\mathrm{d}y. \tag{35}$$

We shall suppose that the eigenvalues are simple and that the eigenfunctions are complete. With these assumptions we can now represent the δ-function (matrix) in terms of the y-eigenfunctions: We have [6, Chapter 4]

$$\begin{pmatrix} \delta(y-y') & 0 \\ 0 & \delta(y-y') \end{pmatrix} = \sum_n \frac{V_n(y)V_n^T(y')\Phi}{N_{yn}}, \tag{36}$$

Table 3. Coefficients $\xi_1,\ \xi_2,\ \eta_1,\ \eta_2$ for various y-boundary conditions; dependence of γ on m and n is suppressed.

Case	ξ_1	ξ_2	η_1	η_2
Y11	$-\cosh(\gamma b)$	$\sinh(\gamma b)$	$-\cosh(\gamma c)$	$\sinh(\gamma c)$
Y12	$-\cosh(\gamma b)$	$\sinh(\gamma b)$	$-\sinh(\gamma c)$	$\cosh(\gamma c)$
Y13	$-\cosh(\gamma b)$	$\sinh(\gamma b)$	$-k_2\gamma\sinh(\gamma c)$ $-h\cosh(\gamma c)$	$k_2\gamma\cosh(\gamma c)$ $+h\sinh(\gamma c)$
Y21	$\sinh(\gamma b)$	$\cosh(\gamma b)$	$-\cosh(\gamma c)$	$\sinh(\gamma c)$
Y22	$\sinh(\gamma b)$	$\cosh(\gamma b)$	$-\sinh(\gamma c)$	$\cosh(\gamma c)$
Y23	$\sinh(\gamma b)$	$\cosh(\gamma b)$	$-k_2\gamma\sinh(\gamma c)$ $-h\cosh(\gamma c)$	$k_2\gamma\cosh(\gamma c)$ $+h\sinh(\gamma c)$
Y31	$k_1\gamma\sinh(\gamma b)$ $-h\cosh(\gamma b)$	$k_1\gamma\cosh(\gamma b)$ $-h\sinh(\gamma b)$	$-\cosh(\gamma c)$	$\sinh(\gamma c)$
Y32	$k_1\gamma\sinh(\gamma b)$ $-h\cosh(\gamma b)$	$k_1\gamma\cosh(\gamma b)$ $-h\sinh(\gamma b)$	$-\sinh(\gamma c)$	$\cosh(\gamma c)$
Y33	$k_1\gamma\sinh(\gamma b)$ $-h\cosh(\gamma b)$	$k_1\gamma\cosh(\gamma b)$ $-h\sinh(\gamma b)$	$-k_2\gamma\sinh(\gamma c)$ $-h\cosh(\gamma c)$	$k_2\gamma\cosh(\gamma c)$ $+h\sinh(\gamma c)$

where N_{yn} and Φ are given by

$$N_{yn} = \langle V_n(y), V_n(y)\rangle, \quad \Phi = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{37}$$

(N_{yn} is the norm squared.) These generalized y-eigenfunctions will be used in the construction of alternative forms of the Green's functions.

5. Construction of the Green's functions of the first form

The first form of the Green's function, $g_{ij}^{(1)}(x, y, z, x', y', z')$ is sought as

$$g_{ij}^{(1)}(\vec{x}, \vec{x}') = \sum_n \sum_m \psi_{mnij}(y, y') \frac{u_m(x)u_m(x')}{N_{xm}} \frac{w_n(z)w_n(z')}{N_{zn}}, \quad i, j = 1, 2, \tag{38}$$

in terms of the x- and z-eigenfunctions. Consider the case $j = 1$ first. Substituting Equation (38) in Equation (5) and using the relation that [6, Chapter 4]

$$\delta(x - x') = \sum_m \frac{u_m(x)u_m(x')}{N_{xm}}, \ \delta(z - z') = \sum_n \frac{w_n(z)w_n(z')}{N_{zn}}, \tag{39}$$

we obtain

$$\frac{\mathrm{d}^2\psi_{11}}{\mathrm{d}y^2} - \gamma_{mn}^2\psi_{11} = -\frac{1}{k_1}\delta(y - y'), \quad -b < y < 0, \tag{40}$$

$$\frac{\mathrm{d}^2\psi_{21}}{\mathrm{d}y^2} - \gamma_{mn}^2\psi_{21} = 0, \ 0 < y < c, \tag{41}$$

where

$$\gamma_{mn}^2 = \beta_m^2 + \nu_n^2. \tag{42}$$

Notice that the dependence of the functions ψ_{1j} and ψ_{2j} on m and n has been suppressed.

For ψ_{11} and ψ_{21} above we have the general solutions

$$\psi_{11}(y, y') = A(y')\sinh(\gamma_{mn}y) + B(y')\cosh(\gamma_{mn}y), -b < y < y', \tag{43}$$

$$\psi_{11}(y, y') = C(y')\sinh(\gamma_{mn}y) + D(y')\cosh(\gamma_{mn}y), y' < y < 0, \tag{44}$$

$$\psi_{21}(y, y') = E(y')\sinh(\gamma_{mn}y) + F(y')\cosh(\gamma_{mn}y), 0 < y < c. \tag{45}$$

The six constants $A, \ldots, F$ are determined from the six conditions below: (i) The boundary conditions on ψ_{11} at $y = -b$ and on ψ_{21} at $y = c$; (ii) The interface conditions involving ψ_{11} and ψ_{21} at $y = 0$; (iii) The continuity of ψ_{11} and jump condition of $\mathrm{d}\psi_{11}/\mathrm{d}y$ at $y = y'$.

Application of the boundary conditions at $y = -b$ and $y = c$ leads to

$$A = \xi_1 G,\ B = \xi_2 G,\ E = \eta_1 H,\ F = \eta_2 H. \tag{46}$$

The ξ's and η's depend on the y-boundary conditions and can be found in Table 2 as

$$\xi_1 = \sinh(\gamma b), \xi_2 = \cosh(\gamma b), \eta_1 = \sinh(\gamma c), \eta_2 = \cosh(\gamma c). \tag{47}$$

We have thus four remaining constants to be determined from the two interface conditions at $y = 0$ and the continuity and the jump conditions at $y = y'$.

The interface conditions relate C, D and E, F by

$$M_1 \begin{pmatrix} C \\ D \end{pmatrix} = M_2 \begin{pmatrix} E \\ F \end{pmatrix}, \tag{48}$$

where

$$M_1 = \begin{pmatrix} k_1\gamma_{mn}/h & 1 \\ 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ \delta & 0 \end{pmatrix}. \tag{49}$$

Thus,

$$\begin{pmatrix} C \\ D \end{pmatrix} = M_1^{-1} M_2 \begin{pmatrix} E \\ F \end{pmatrix} = M_1^{-1} M_2 M_5 H,\quad M_3 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{50}$$

We can thus write

$$C = \alpha_1 H, D = \alpha_2 H, \tag{51}$$

where α_1 and α_2 are the first and the second elements, respectively, of $M_1^{-1} M_2 M_5$ and are given by

$$\alpha_1 = -\delta\sinh(\gamma_{mn}c),\quad \alpha_2 = \cosh(\gamma_{mn}c) + \delta k_1\gamma_{mn}\sinh(\gamma_{mn}c)/h. \tag{52}$$

We can now write

$$\psi_{11}(y, y') = \begin{cases} G(y')q_1(y) & -b < y < y', \\ H(y')q_2(y) & y' < y < 0 \end{cases} \tag{53}$$

where

$$q_1(y) = \xi_1 \sinh(\gamma_{mn} y) + \xi_2 \cosh(\gamma_{mn} y), \tag{54}$$
$$q_2(y) = \alpha_1 \sinh(\gamma_{mn} y) + \alpha_2 \cosh(\gamma_{mn} y). \tag{55}$$

Applying the continuity and jump conditions of ψ_{11} and $\mathrm{d}\psi_{11}/\mathrm{d}y$ at $y = y'$ now leads to

$$G(y') = -(1/(k_1 W_1))q_2(y'),\ H(y') = -(1/(k_1 W_1))q_1(y'), \tag{56}$$

where W_1 is the Wronskian of $q_1(y)$ and $q_2(y)$ and is given by

$$W_1 = -\gamma_{mn}(\delta \cosh(\gamma_{mn} b) \sinh(\gamma_{mn} c) + \sinh(\gamma_{mn} b) \cosh(\gamma_{mn} c)) + \delta k_1 \gamma_{mn} \sinh(\gamma_{mn} b) \sinh(\gamma_{mn} c)/h. \tag{57}$$

Substituting G in Equations (46) and H in (46) and (51) we may determine A, B, C, D, E and F.

For ψ_{12} and ψ_{22} we have

$$\frac{\mathrm{d}^2\psi_{12}}{\mathrm{d}y^2} - \gamma_{mn}^2 \psi_{12} = 0,\ 0 < y < b, \tag{58}$$
$$\frac{\mathrm{d}^2\psi_{22}}{\mathrm{d}y^2} - \gamma_{mn}^2 \psi_{22} = -\frac{1}{k_2}\delta(y - y'),\ b < y < c. \tag{59}$$

Thus,

$$\psi_{12}(y, y') = \tilde{A}(y') \sinh(\gamma_{mn} y) + \tilde{B}(y') \cosh(\gamma_{mn} y)\ -b < y < 0, \tag{60}$$
$$\psi_{22}(y, y') = \tilde{C}(y') \sinh(\gamma_{mn} y) + \tilde{D}(y') \cosh(\gamma_{mn} y)\ 0 < y < y', \tag{61}$$
$$\psi_{22}(y, y') = \tilde{E}(y') \sinh(\gamma_{mn} y) + \tilde{F}(y') \cosh(\gamma_{mn} y)\ y' < y < c. \tag{62}$$

Applying the boundary conditions at $y = 0$ and $y = c$ we have

$$\tilde{A} = \xi_1 \tilde{G},\ \tilde{B} = \xi_2 \tilde{G},\ \tilde{E} = \eta_1 \tilde{H},\ \tilde{F} = \eta_2 \tilde{H}. \tag{63}$$

The four remaining constants are determined by using the two interface conditions involving ψ_{21} and ψ_{22} at $y = 0$ and the continuity of ψ_{22} and jump condition of $\mathrm{d}\psi_{22}/\mathrm{d}y$ at $y = y'$. Omitting the details we have

$$\psi_{22}(y.y') = \begin{cases} \tilde{G}(y')q_3(y) & b < y < y', \\ \tilde{H}(y')q_4(y) & y' < y < c, \end{cases} \tag{64}$$

where

$$q_3(y) = \beta_1 \sinh(\gamma_{mn} y) + \beta_2 \cosh(\gamma_{mn} y), \tag{65}$$
$$q_4(y) = \eta_1 \sinh(\gamma_{mn} y) + \eta_2 \cosh(\gamma_{mn} y). \tag{66}$$

and

$$\beta_1 = (1/\delta) \sinh(\gamma_{mn} b),\ \beta_2 = \cosh(\gamma_{mn} b) + k_1 \gamma_{mn} \sinh(\gamma_{mn} c)/h. \tag{67}$$

$\tilde{G}(y')$ and $\tilde{H}(y')$ are given by

$$\tilde{G}(y') = -\frac{1}{k_2 W_2} q_4(y'),\ \tilde{H}(y') = -\frac{1}{k_2 W_2} q_3(y'), \tag{68}$$

where W_2 is the Wronskian of $q_3(y)$ and $q_4(y)$ and is given by

$$\begin{aligned} W_2 &= -\gamma_{mn}(\frac{1}{\delta}\sinh(\gamma_{mn}b)\cosh(\gamma_{mn}c) + \cosh(\gamma_{mn}b)\sinh(\gamma_{mn}c)) \\ &\quad - \delta k_1\gamma_{mn}\sinh(\gamma_{mn}b)\cosh(\gamma_{mn}c)/h. \end{aligned} \tag{69}$$

6. Solutions for problem A of the first form

In Equations (14) and (15) of Section 3 we have given a representation for the solutions of Problem A of the first form in terms of the Green's functions and the boundary data. Substituting the Green's function given in Equation (38) in Equations (14) and (15), we obtain

$$\begin{aligned} T_1(x,y,z) &= q_0\sum_n\sum_m\int_0^a \frac{u_m(x)u_m(x')}{N_{xm}}\frac{w_n(z)w_n(z')}{N_{zn}}\Big|_{z'=d}\mathrm{d}x' h_1(y) \\ &= \sum_n\sum_m \frac{u_m(x)w_n(z)}{N_{xm}N_{zn}}\frac{(-1)^{n+1}}{\beta_m}h_1(y), \end{aligned} \tag{70}$$

where

$$\begin{aligned} h_1(y) &= \int_{-b}^0 \psi_{11}(y',y)\mathrm{d}y' + \int_0^c \psi_{21}(y',y)\mathrm{d}y' \\ &= \frac{1}{k_1\gamma_{mn}^2} + \frac{1}{k_1W_1}\left(\frac{\delta}{\gamma_{mn}} - \frac{1}{\gamma_{mn}}\right)\sinh(\gamma_{mn}c)\cosh(\gamma_{mn}(y+b)), \end{aligned} \tag{71}$$

$$\begin{aligned} T_2(x,y,z) &= q_0\sum_n\sum_m\int_0^a \frac{u_m(x)u_m(x')}{N_{xm}}\frac{w_n(z)w_n(z')}{N_{zn}}\Big|_{z'=d}\mathrm{d}x' h_2(y) \\ &= \sum_n\sum_m \frac{u_m(x)w_n(z)}{N_{xm}N_{zn}}\frac{(-1)^{n+1}}{\beta_m}h_2(y), \end{aligned} \tag{72}$$

$$\begin{aligned} h_2(y) &= \int_{-b}^0 \psi_{12}(y',y)\mathrm{d}y' + \int_0^c \psi_{22}(y',y)\mathrm{d}y' \\ &= \frac{1}{k_2\gamma_{mn}^2} - \frac{1}{k_2W_2}\left(\frac{1}{\gamma_{mn}} - \frac{1}{\delta\gamma_{mn}}\right)\sinh(\gamma_{mn}b)\cosh(\gamma_{mn}(y-c)). \end{aligned} \tag{73}$$

Notice that the dependence of $h_1(y)$ and $h_2(y)$ on m and n has been suppressed. The term $1/(k_1\gamma_{mn}^2)$ in $h_1(y)$ and the term $1/(k_2\gamma_{mn}^2)$ in $h_2(y)$ lead to slowly convergent series as the terms decay only algebraically for large m and n while the remaining terms in h_1 and h_2 decay exponentially for large m and n for y not equal to $-b$ or c, owing to the hyperbolic factors in W_1 and W_2 in their denominators. We recognize that these slowly convergent series, which are independent of y, are solutions of two-dimensional boundary-value problems $\tilde{T}_i(x,z)$ with the boundary data:

$$\tilde{T}_i(0,z) = 0,\ \tilde{T}_i(x,0) = 0,\ \frac{\partial\tilde{T}_i}{\partial x}(a,z) = 0,\ k_i\frac{\partial\tilde{T}_i}{\partial z}(x,d) = q_0. \tag{74}$$

This can be verified by showing that the double sum

$$\tilde{T}_i(x,z) = q_0\sum_n\sum_m \frac{u_m(x)w_n(z)}{N_{xm}N_{zn}}\frac{(-1)^{n+1}}{\beta_n}\frac{1}{k_i\gamma_{mn}^2}, \tag{75}$$

is the representation of the solution of the two-dimensional boundary-value problem using the double sum Green's function

$$g_i(x, z, x', z') = \sum_n \sum_m \frac{u_m(x)u_m(x')}{N_{xm}} \frac{w_n(z)w_n(z')}{N_{zn}} \frac{1}{k_i \gamma_{mn}^2}. \tag{76}$$

These slowly convergent series given by $\tilde{T}_i(x, z)$ can be further simplified by casting them as single sums using the single sum Green's functions. Omitting the details, we have the following results:

$$\tilde{T}_i(x, z) = q_0 \sum_m \frac{u_m(x)}{N_{xm}} \frac{1}{k_i \beta_m^2} \frac{\sinh(\beta_m z)}{\cosh(\beta_m d)}. \tag{77}$$

and

$$\tilde{T}_i(x, z) = q_0 \sum_n (-1)^{(n+1)} \frac{w_n(z)}{N_{zn}} \frac{-\cosh(\nu_n(x-a)) + \cosh(\nu_n a)}{k_i \nu_n^2 \cosh(\nu_n a)}. \tag{78}$$

These expressions for $\tilde{T}_i$ converge rapidly for $z \neq d$ and $x \neq 0$, respectively.

7. Solutions for problem A using alternative Green's functions

We now construct alternative Green's functions and then use them to obtain alternative solutions for Problem A. We take

$$g_{ij}^{(2)}(x, y, z, x', y', z') = \sum_n \sum_m \alpha_{mn}(z, z') \frac{u_m(x)u_m(x')}{N_{xm}} \frac{v_{ni}(y)v_{nj}(y')}{N_{yn}} \tag{79}$$

where $\alpha_{mn}(z, z')$ satisfies

$$\frac{d^2}{dz^2}\alpha_{mn}(z) - \rho_{mn}^2 \alpha_{mn}(z) = -\delta(z - z'), \ \rho_{mn}^2 = \beta_m^2 + \mu_n^2 \tag{80}$$

and the boundary conditions $\alpha_{mn}(0) = \alpha'_{mn}(d) = 0$ and is given by

$$\alpha_{mn}(z, z') = \begin{cases} \dfrac{\cosh(\rho_{mn}(d - z'))\sinh(\rho_{mn} z)}{\rho_{mn}\cosh(\rho_{mn} d)}, & 0 < z < z', \\ \dfrac{\sinh(\rho_{mn} z')\cosh(\rho_{mn}(d - z))}{\rho_{mn}\cosh(\rho_{mn} d)}, & z' < z < d. \end{cases} \tag{81}$$

We substitute the above in Equations (14) and (15) and, taking $k_i \frac{\partial T_i}{\partial z'}|_{z'=d} = q_0, i = 1, 2,$ we obtain

$$\begin{aligned} T_1^{(2A)}(x, y, z) &= q_0 \sum_{n=0} \sum_{m=1} \alpha_{mn}(z', z)|_{z'=d} \frac{u_m(x)}{N_{xm}} \int_0^a u_m(x')dx' \frac{v_{n1}(y)}{N_{yn}} \\ &\quad \times \left[\int_{-b}^0 v_{n1}(y')dy' + \int_0^c v_{n2}(y')dy' \right] \\ &= q_0 \sum_{m=1} \frac{\sinh(\rho_{m0} z)}{rho_{m0}\cosh(\rho_{m0} d)} \frac{1}{\beta_m} \frac{u_m(x)}{N_{xm}} \frac{v_{01}(y)}{N_{y0}} I_0 \end{aligned}$$

$$+ q_0 \sum_{n=1} \sum_{m=1} \frac{\sinh(\rho_{mn} z)}{\rho_{mn} \cosh(\rho_{mn} d)} \frac{1}{\beta_m} \frac{u_m(x)}{N_{xm}} \frac{v_{n1}(y)}{N_{yn}} I_n$$

$$= q_0 \sum_{m=1} \frac{\sinh(\beta_m z)}{\beta_m^2 \cosh(\beta_m d)} \frac{u_m(x)}{N_{xm}} \frac{b+c}{k_1 b + k_2 c}$$

$$+ q_0 \sum_{n=1} \sum_{m=1} \frac{\sinh(\rho_{mn} z)}{\beta_m \rho_{mn} \cosh(\rho_{mn} d)} \frac{u_m(x)}{N_{xm}} \frac{v_{n1}(y)}{N_{yn}} I_n \quad (82)$$

$$T_2^{(2A)}(x, y, z) = q_0 \sum_{m=1} \frac{\sinh(\beta_m z)}{\beta_m^2 \cosh(\beta_m d)} \frac{u_m(x)}{N_{xm}} \frac{b+c}{k_1 b + k_2 c}$$

$$+ q_0 \sum_{n=1} \sum_{m=1} \frac{\sinh(\rho_{mn} z)}{\beta_m \rho_{mn} \cosh(\rho_{mn} d)} \frac{u_m(x)}{N_{xm}} \frac{v_{n2}(y)}{N_{yn}} I_n \quad (83)$$

where

$$I_n \int_{-b}^{0} v_{n1}(y')\mathrm{d}y' + \int_{0}^{c} v_{n2}(y')\mathrm{d}y', \text{‘} n = 0, 1, ... \quad (84)$$

We consider now the case when the Green's function is expanded in terms of the y- and z-eigenfunctions.

$$g_{ij}^{(3)}(x, y, z, x', y', z') = \sum_n \sum_m \phi_{mn}(x, x') \frac{w_m(z) w_m(z')}{N_{zm}} \frac{v_{ni}(y) v_{nj}(y')}{N_{yn}} \quad (85)$$

Proceeding as in the previous case, we obtain the solutions

$$T_1^{(3A)}(x, y, z) = q_0 \sum_n \sum_m \left(\frac{1}{\lambda_{mn}^2} - \frac{\cosh(\lambda_{mn}(a-x))}{\lambda_{mn}^2 \cosh(\lambda_{mn} a)} \right) \frac{w_m(z)}{N_{zm}} \times (-1)^{(m+1)} \frac{v_{n1}(y)}{N_{yn}} I_n \quad (86)$$

$$T_2^{(3A)}(x, y, z) = q_0 \sum_n \sum_m \left(\frac{1}{\lambda_{mn}^2} - \frac{\cosh(\lambda_{mn}(a-x))}{\lambda_{mn}^2 \cosh(\lambda_{mn} a)} \right) \frac{w_m(z)}{N_{zm}} \times (-1)^{(m+1)} \frac{v_{n2}(y)}{N_{yn}} I_n \quad (87)$$

8. Discussion

We have considered an example problem as described in Section 2 and solved it by the method of Green's functions. Green's functions were constructed by using eigenfunctions in two of the three directions. This results in Green's functions of three different forms and leads to different representation of solutions as double sums in terms of the two sets of eigenfunctions chosen in the expansions. The expansion coefficients depend on the third variable and the components in the third variable, known as the 'kernels', involve in general quotients of hyperbolic functions that decay exponentially for the solutions away from possibly edges of the boundary or the interface. The different solutions T_i^{1A}, T_i^{2A}, and T_i^{3A} have different convergence characteristics and a judicious selection of them for a given computation is important. As we pointed out in the paper, there are also cases involving lower-dimensional sums where the decay is only algebraic. These cases can usually be identified and summed in closed forms.

We present in Table 4 numerical results for the temperature T_1 at selected points (x, y, z) as listed in columns 1 through 3, by using the expressions for $T_1^{(1A)}$, $T_1^{(2A)}$ and $T_1^{(3A)}$. We use

Table 4. Temperatures for material 1 using the three solutions of problem A. $a = 1$, $d = 1$, $b = c = 0{\cdot}25$. Various values of x, y and z. $k_2/k_1 = 2{\cdot}0$. Maximum of 500 terms in a given summation index. Underlining indicates inaccurate digits.

$\frac{x}{a}$	$\frac{y}{b}$	$\frac{z}{d}$	$\frac{T_1^{(1A)}(x,y,z)}{q_0 d/k_1}$	# terms (1A)	$\frac{T_1^{(2A)}(x,y,z)}{q_0 d/k_1}$	# terms (2A)	$\frac{T_1^{(3A)}(x,y,z)}{q_0 d/k_1}$	# terms (3A)
0·0	0·0	0·5	−0·0000005709	250001	0·0000000000	192	0·0000000000	15
0·2	0·0	0·5	0·0628361729	250001	0·0628361726	192	0·0628361726	1337
0·4	0·0	0·5	0·1152792007	250001	0·1152792003	192	0·1152792003	1674
0·6	0·0	0·5	0·1526840845	250001	0·1526840841	192	0·1526840841	1826
0·8	0·0	0·5	0·1746289296	250001	0·1746289292	192	0·1746289292	1913
1·0	0·0	0·5	0·1818243110	250001	0·1818243107	192	0·1818243107	1969
0·0	0·0	1·0	0·0004044756	250001	0·0000000000	250501	0·0000000000	500
0·2	0·0	1·0	0·2280601069	250001	0·2279239192	250501	0·2280601591	1822
0·4	0·0	1·0	0·3382522706	250001	0·3381166011	250501	0·3382524264	2159
0·6	0·0	1·0	0·4036678995	250001	0·4035323872	250501	0·4036680868	2311
0·8	0·0	1·0	0·4390871545	250001	0·4389517045	250501	0·4390873542	2398
1·0	0·0	1·0	0·4503448180	250001	0·4502093854	250501	0·4503450211	2454
0·0	−1·0	0·5	−0·0000005709	649	0·0000000000	192	0·0000000000	15
0·2	−1·0	0·5	0·0642905850	649	0·0642905850	192	0·0642905850	1337
0·4	−1·0	0·5	0·1176181339	649	0·1176181339	192	0·1176181339	1674
0·6	−1·0	0·5	0·1554013140	649	0·1554013140	192	0·1554013140	1826
0·8	−1·0	0·5	0·1774761213	649	0·1774761213	192	0·1774761213	1913
1·0	−1·0	0·5	0·1847012264	649	0·1847012264	192	0·1847012264	1969
0·0	−1·0	1·0	0·0004044756	649	0·0000000000	250501	0·0000000000	30
0·2	−1·0	1·0	0·2796425639	649	0·2796412188	250501	0·2796425639	1352
0·4	−1·0	1·0	0·3976833881	649	0·3976826649	250501	0·3976833881	1689
0·6	−1·0	1·0	0·4648386905	649	0·4648381558	250501	0·4648386905	1841
0·8	−1·0	1·0	0·5006706168	649	0·5006701569	250501	0·5006706168	1928
1·0	−1·0	1·0	0·5120066012	649	0·5120061622	250501	0·5120066012	1984

the expressions for closed-form sums for the 2-D components in such expressions whenever we can to obtain the total temperatures. We aim at solutions for the temperatures, normalized with respect to $q_0 d/k_1$ with at least ten-place accuracy so that the solutions may be used for purposes of verification of computer codes. The temperatures are given in columns 4, 6 and 8, while the corresponding maximum numbers of terms used in the calculations are given in columns 5, 7 and 9. Results not achieving this accuracy are underlined. It is seen that there are inaccurate results in Table 4 that occur at $x = 0$ and at $z = d$. The different representation of solutions does, however, complement one another when highly accurate numerical results are desired. The method presented here can treat a large class of boundary-value problems similar to Problem A.

Numerical results for the temperature in layer 2 behave similarly to those obtained here for layer 1 and are omitted.

We conclude this paper by making some observations on the method of Green's function and the classical method of separation of variables. Both methods, when same sets of eigenfunctions are used, lead to similar results. However, they differ in the way as they are implemented. To see this difference let us consider a simple rectangular region in two dimensions instead of a three-dimensional parallelepiped. Let the region be $0 < x < a$, $0 < y < b$. Suppose that homogeneous boundary data are given along $x = 0$ and $x = a$. One can readily obtain solution using separation of variables in the form of expansion in terms of eigenfunctions $(u_n(x))$. Now suppose that non-homogeneous boundary data are given along $x = 0$ and $x = a$. This problem has two forms of Green's functions, and likewise there are two ways to do the separation of variables. The non-homogeneous boundary conditions along $x = 0$ and $x = a$ discourages expansion of the solution in terms of the x-eigenfunctions, though it can still be done but with some extra work. We introduce lower-dimensional solution terms to transfer the non-homogeneous data to the boundary $y = 0$ and $y = b$ so that the separation of variables and the x-expansion can work. The method of Green's functions does not suffer this limitation; it automatically finds this 'transfer' function that serves to move the non-homogeneous data from one boundary to another. It should also be pointed out that in the simple problem above the solution obtained using the x-expansion has the kernel in the y-direction resulting in fast decay of the solution in the y-direction, a feature that we need when we wish to compute solutions on the boundaries $x = 0$ or $x = a$.

Acknowledgements

We appreciate the support of this research by Sandia National Laboratories, Albuquerque, New Mexico. Dr. Kevin J. Dowding was the project manager at Sandia.

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